

ON f -DERIVATIONS OF BE-ALGEBRAS

KYUNG HO KIM* AND B. DAVVAZ**

ABSTRACT. In this paper, we introduce the notion of f -derivation in a BE- algebra, and consider the properties of f -derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by f -derivations. Moreover, we prove that if d is a f -derivation of a BE-algebra, every f -filter F is a d -invariant.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. The notion of a BE-algebra is a dualization of a generalization of a BCK-algebra. In this paper, we introduce the notion of f -derivation in a BE- algebra, and consider the properties of f -derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by f -derivations. Moreover, we prove that if d is a f -derivation of a BE-algebra, every f -filter F is a d -invariant.

2. Preliminaries

In what follows, let X denote an BE-algebra unless otherwise specified.

By a BE-algebra we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation “ $*$ ” that satisfies the following identities: for any $x, y, z \in X$,

(BE1) $x * x = 1$ for all $x \in X$,

Received November 19, 2014; Accepted January 27, 2015.

2010 Mathematics Subject Classification: Primary 06F35, 03G25, 08A30.

Key words and phrases: BE-algebra, self-distributive, filter(normal filter), derivation, f -derivation, isotone, $Kerd$.

- (BE2) $x * 1 = 1$ for all $x \in X$,
 (BE3) $1 * x = x$ for all $x \in X$,
 (BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra $(X, *, 1)$ is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. For any x, y in a BE-algebra X , we define $x \vee y = (y * x) * x$.

In a BE-algebra, the following identities are true: for any $x, y, z \in X$,

- (p1) $x * (y * x) = 1$.
 (p2) $x * ((x * y) * y) = 1$.
 (p3) Let X be a self-distributive BE-algebra. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.

DEFINITION 2.1. A non-empty subset F of X is called a *filter* of X if

- (F1) $1 \in F$,
 (F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

DEFINITION 2.2. Let X be a BE-algebra. We say that X is *commutative* if

$$(x * y) * y = (y * x) * x$$

for all $x, y \in X$.

DEFINITION 2.3. A self-map d on a BE-algebra X is called a *derivation* if

$$d(x * y) = (x * d(y)) \vee (d(x) * y)$$

for every $x, y \in X$.

EXAMPLE 2.4. Let $X = \{1, a, b\}$ be a set in which “*” is defined by

$$\begin{array}{c|ccc} * & 1 & a & b \\ \hline 1 & 1 & a & b \\ a & 1 & 1 & b \\ b & 1 & a & 1 \end{array}$$

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ b & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a derivation of a BE-algebra X .

DEFINITION 2.5. A self-map d on a BE-algebra X is called to be *regular* if $d(1) = 1$.

DEFINITION 2.6. Let X be a BE-algebra. We define the binary operation “ \leq ” as the following,

$$x \leq y \Leftrightarrow x * y = 1$$

for all $x, y \in X$.

3. f -derivations of BE-algebras

DEFINITION 3.1. Let X be a BE-algebra. A function $d : X \rightarrow X$ is called an f -derivation on X if there exists an endomorphism $f : X \rightarrow X$ such that

$$d(x * y) = (f(x) * d(y)) \vee (d(x) * f(y))$$

for every $x, y \in X$.

EXAMPLE 3.2. Let $X = \{1, a, b\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ b & \text{if } x = a \end{cases}$$

and define an endomorphism $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a, b \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X .

EXAMPLE 3.3. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

and define an endomorphism $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a \\ b & \text{if } x = c \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X .

EXAMPLE 3.4. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ c & \text{if } x = a, c \end{cases}$$

and define an endomorphism $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = a, c \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X .

EXAMPLE 3.5. Let $X = \{1, a, b\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ a & \text{if } x = b \end{cases}$$

and define an endomorphism $f : X \rightarrow X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, a \\ b & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a f -derivation of a BE-algebra X . But d is not a derivation of X since

$$\begin{aligned} a &= d(b) = d(a * b) \neq (a * d(b)) \vee (d(a) * b) \\ &= (a * a) \vee (1 * b) = 1 \vee b = (b * 1) * 1 = 1 * 1 = 1. \end{aligned}$$

PROPOSITION 3.6. *Every endomorphism f of a BE-algebra X is its f -derivation.*

Proof. Let X be a BE-algebra and let f be an endomorphism on X . Then

$$f(x) * f(y) \vee f(x) * f(y) = f(x) * f(y) = f(x * y)$$

for all $x, y \in X$. This completes the proof. \square

PROPOSITION 3.7. *Let X be a BE-algebra. Then every f -derivation of X is regular.*

Proof. Since f is an endomorphism on X , we have $f(1) = 1$. Hence we have

$$\begin{aligned} d(1) &= d(x * 1) = (f(x) * d(1)) \vee (d(x) * f(1)) \\ &= (f(x) * d(1)) \vee (d(x) * 1) \\ &= (f(x) * d(1)) \vee 1 \\ &= 1. \end{aligned}$$

This completes the proof. \square

PROPOSITION 3.8. *Let X be a BE-algebra and let d be a f -derivation on X . Then $d(x) = d(x) \vee f(x)$ for all $x \in X$.*

Proof. For all $x \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = (f(1) * d(x)) \vee (d(1) * f(x)) \\ &= (f(1) * d(x)) \vee (1 * f(x)) = (1 * d(x)) \vee f(x) \\ &= d(x) \vee f(x). \end{aligned}$$

\square

PROPOSITION 3.9. *Let X be a BE-algebra. If d is a f -derivation of X , then the following identities hold:*

- (1) $f(x) \leq d(x)$ for all $x \in X$,
- (2) $d(x) * f(y) \leq f(x) * d(y)$ for all $x, y \in X$.

Proof. (1) By Proposition 3.8, we have

$$\begin{aligned} f(x) * d(x) &= f(x) * (d(x) \vee f(x)) = f(x) * ((f(x) * d(x)) * d(x)) \\ &= (f(x) * d(x)) * (f(x) * d(x)) \\ &= 1 \end{aligned}$$

which implies $f(x) \leq d(x)$.

(2) From (1) and (p3), we have $d(x) * f(y) \leq f(x) * f(y) \leq f(x) * d(y)$. \square

THEOREM 3.10. *Let X be a BE-algebra and let d be a f -derivation of X . Then we have $d(x * y) = f(x) * d(y)$ for all $x, y \in X$.*

Proof. Let d be a f -derivation on X and $x, y \in X$. Then we have $d(x) * f(y) \leq f(x) * d(y)$ from Proposition 3.9 (2). Hence we get

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= ((d(x) * f(y)) * (f(x) * d(y))) * (f(x) * d(y)) \\ &= 1 * (f(x) * d(y)) = f(x) * d(y). \end{aligned}$$

\square

PROPOSITION 3.11. *Let X be a BE-algebra and let d be a f -derivation of X . If it satisfies $d(x * y) = d(x) * f(y)$ for all $x, y \in X$, we have $d(x) = f(x)$.*

Proof. Let d be a f -derivation of X . If it satisfies $d(x * y) = d(x) * f(y)$ for all $x, y \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = d(1) * f(x) \\ &= 1 * f(x) = f(x). \end{aligned}$$

This completes the proof. \square

PROPOSITION 3.12. *Let X be a BE-algebra and let d be a f -derivation of X . If it satisfies $f(x) * d(y) = d(x) * f(y)$ for all $x, y \in X$, then $d(x) = f(x)$ for all $x \in X$.*

Proof. Let d be a f -derivation of X . If it satisfies $f(x) * d(y) = d(x) * f(y)$ for all $x, y \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = f(1) * d(x) \\ &= d(1) * f(x) = 1 * f(x) \\ &= f(x) \end{aligned}$$

from Theorem 3.10. This completes the proof. \square

THEOREM 3.13. *Let d be a f -derivation on X . If $d \circ f = f \circ d$, then we have $d(f(x) * d(x)) = 1$ for all $x \in X$.*

Proof. Let d be a f -derivation on X and $d \circ f = f \circ d$. For all $x \in X$, we have

$$\begin{aligned} d(f(x) * d(x)) &= (f(f(x)) * d(d(x))) \vee (d(f(x)) * f(d(x))) \\ &= (f(f(x)) * d(d(x))) \vee (f(d(x)) * f(d(x))) \\ &= (f(f(x)) * d(d(x))) \vee 1 = 1. \end{aligned}$$

□

DEFINITION 3.14. Let X be a BE-algebra and let d be a f -derivation on X . If $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in X$, then d is called an *isotone f -derivation* of X .

PROPOSITION 3.15. *Let d be a f -derivation of a BE-algebra X . If $d(x) \vee d(y) \leq d(x \vee y)$ for all $x, y \in X$, then d is an isotone f -derivation of X .*

Proof. Suppose that $d(x) \vee d(y) \leq d(x \vee y)$ and $x \leq y$. Then we have $d(x) \leq d(x) \vee d(y) \leq d(x \vee y) = d(y)$. □

Let d be a f -derivation of X . Define a set $Fix_d(X)$ by

$$Fix_d(X) := \{x \in X \mid d(x) = f(x)\}$$

for all $x \in X$.

PROPOSITION 3.16. *Let d be a f -derivation of a BE-algebra X . Then $Fix_d(X)$ is a subalgebra of X .*

Proof. Clearly, $1 \in Fix_d(X)$ and so $Fix_d(X)$ is non-empty. Let $x, y \in Fix_d(X)$. Then we have $d(x) = f(x)$ and $d(y) = f(y)$, and so

$$d(x * y) = (f(x) * d(y)) \vee (d(x) * f(y)) = (f(x) * f(y)) \vee (f(x) * f(y)) = f(x * y).$$

This implies $x * y \in Fix_d(X)$. □

PROPOSITION 3.17. *Let X be a BE-algebra and let d be a f -derivation of X . If $x, y \in Fix_d(X)$, then we have $x \vee y \in Fix_d(X)$.*

Proof. Let $x, y \in Fix_d(X)$. Then we have $d(x) = f(x)$ and $d(y) = f(y)$, and so

$$\begin{aligned}
d(x \vee y) &= d((y * x) * x) = (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\
&= ((f(y) * f(x)) * f(x)) \vee (((f(y) * d(x)) \vee (d(y) * f(x))) * f(x)) \\
&= (f(y) * f(x)) * f(x) \vee ((f(y) * f(x)) \vee (f(y) * f(x))) * f(x) \\
&= (f(y) * f(x)) * f(x) \vee ((f(y) * f(x)) * f(x)) \\
&= ((f(y) * f(x)) * f(x)) = f((y * x) * x) = f(x \vee y).
\end{aligned}$$

This completes the proof. \square

Let d be a f -derivation of X . Define a $Kerd$ by

$$Kerd = \{x \in X \mid d(x) = 1\}$$

for all $x \in X$.

PROPOSITION 3.18. *Let d be a f -derivation of X . Then $Kerd$ is a subalgebra of X .*

Proof. Clearly, $1 \in Kerd$, and so $Kerd$ is non-empty. Let $x, y \in Kerd$. Then $d(x) = 1$ and $d(y) = 1$. Hence we have

$$\begin{aligned}
d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\
&= (f(x) * 1) \vee (1 * f(y)) = 1 \vee f(y) \\
&= (f(y) * 1) * 1 = 1 * 1 \\
&= 1,
\end{aligned}$$

and so $x * y \in Kerd$. Thus $Kerd$ is a subalgebra of X . \square

PROPOSITION 3.19. *Let X be a commutative BE-algebra and let d be a f -derivation of X . If $x \in Kerd$ and $x \leq y$, then we have $y \in Kerd$.*

Proof. Let $x \in Kerd$ and $x \leq y$. Then $d(x) = 1$ and $x * y = 1$.

$$\begin{aligned}
d(y) &= d(1 * y) = d((x * y) * y) \\
&= d((y * x) * x) \\
&= (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\
&= (f(y * x) * 1) \vee (d(y * x) * f(x)) \\
&= 1 \vee (d(y * x) * f(x)) \\
&= 1,
\end{aligned}$$

and so $y \in Kerd$. This completes the proof. \square

PROPOSITION 3.20. *Let X be a BE-algebra and let d be a f -derivation of X . If $y \in Kerd$, then we have $x * y \in Kerd$ for all $x \in X$.*

Proof. Let $y \in \text{Kerd}$. Then $d(y) = 1$. Thus we have

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (d(x) * f(y)) \\ &= 1 \vee (d(x) * f(y)) \\ &= 1, \end{aligned}$$

which implies $x * y \in \text{Kerd}$. \square

PROPOSITION 3.21. *Let X be a BE-algebra and let d be a f -derivation of X . If $x \in \text{Kerd}$, then we have $x \vee y \in \text{Kerd}$ for all $y \in X$.*

Proof. Let $x \in \text{Kerd}$. Then $d(x) = 1$. Then we have

$$\begin{aligned} d(x \vee y) &= d((y * x) * x) = (f(y * x) * d(x)) \vee (d(y * x) * f(x)) \\ &= (f(y * x) * 1) \vee (d(y * x) * f(x)) \\ &= 1 \vee (d(y * x) * f(x)) \\ &= 1, \end{aligned}$$

which implies $x \vee y \in \text{Kerd}$. \square

PROPOSITION 3.22. *Let X be a BE-algebra and let d be a f -derivation. If d is an endomorphism on X , then Kerd is a filter of X .*

Proof. Clearly, $1 \in \text{Kerd}$. Let $x, x * y \in \text{Kerd}$, respectively. Then $d(x) = 1$ and $d(x * y) = 1$. Thus, we have $1 = d(x * y) = d(x) * d(y) = 1 * d(y) = d(y)$, which implies $y \in \text{Kerd}$. This completes the proof. \square

Let X be a BE-algebra. We define the binary operation " + " as the following

$$x + y = (x * y) * y$$

for all $x, y \in X$. Clearly, X is a commutative BE-algebra if and only if $x + y = y + x$ for all $x, y \in X$.

PROPOSITION 3.23. *Let X be a commutative BE-algebra. Then the followings hold for all $x, y, z \in X$,*

- (1) $x + 1 = 1 + x$.
- (2) $x + y = y + x$.
- (3) $x + (y + z) = (x + y) + z$.
- (4) $x + x = x$.

Proof. The proof is clear. \square

PROPOSITION 3.24. *Let X be a BE-algebra and let d be a f -derivation of X . Then the followings hold for all $x, y \in X$,*

- (1) $d(x + 1) = 1$.
 (2) $d(x + x) = f(x) + d(x)$.
 (3) $d(x) = f(x) + d(x)$.

Proof. (1) Let $x \in X$. Then we get

$$\begin{aligned} d(x + 1) &= d((x * 1) * 1) = (f(x * 1) * d(1)) \vee (d(x * 1) * f(1)) \\ &= (f(1) * d(1)) \vee (1 * 1) = 1 \vee 1 = (1 * 1) * 1 \\ &= 1. \end{aligned}$$

(2) Let $x \in X$. Then we have

$$\begin{aligned} d(x + x) &= d((x * x) * x) = (f(x * x) * d(x)) \vee (d(x * x) * f(x)) \\ &= (f(1) * d(x)) \vee (d(1) * f(x)) = (1 * d(x)) \vee (1 * f(x)) \\ &= d(x) \vee f(x) \\ &= (f(x) * d(x)) * d(x) = f(x) + d(x). \end{aligned}$$

(3) Let $x \in X$. Then we have

$$\begin{aligned} d(x) &= d(1 * x) = (f(1) * d(x)) \vee (d(1) * f(x)) \\ &= (1 * d(x)) \vee (1 * f(x)) = d(x) \vee f(x) \\ &= (f(x) * d(x)) * d(x) \\ &= f(x) + d(x) \end{aligned}$$

□

DEFINITION 3.25. Let X be a BE-algebra. A non-empty set F of X is called a *normal filter* of X if it satisfies the following conditions:

- (NF1) $1 \in F$,
 (NF2) $x \in X$ and $y \in F$ imply $x * y \in F$.

EXAMPLE 3.26. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

Then X is a BE-algebra. Let $F = \{1, a\}$. Then F is a normal filter of X .

PROPOSITION 3.27. Let X be a BE-algebra and let d be a f -derivation of X . Then $Fix_d(X)$ is a normal filter of X .

Proof. Clearly, $1 \in \text{Fix}_d(X)$. Let $x \in X$ and $y \in \text{Fix}_d(X)$. Then we have $d(y) = f(y)$, and so

$$\begin{aligned} d(x * y) &= f(x) * d(y) \\ &= f(x) * f(y) \\ &= f(x * y), \end{aligned}$$

which implies $x * y \in \text{Fix}_d(X)$ from Theorem 3.10. This completes the proof. \square

PROPOSITION 3.28. *Let X be a BE-algebra and let d be a f -derivation of X . Then $\text{Ker}d$ is a normal filter of X .*

Proof. Clearly, $1 \in \text{Ker}d$. Let $x \in X$ and $y \in \text{Ker}d$. Then we have $d(y) = 1$, and so

$$\begin{aligned} d(x * y) &= (f(x) * d(y)) \vee (d(x) * f(y)) \\ &= (f(x) * 1) \vee (d(x) * f(y)) \\ &= 1 \vee (d(x) * f(y)) = 1, \end{aligned}$$

which implies $x * y \in \text{Ker}d$. Hence $\text{Ker}d$ is a normal filter of X . \square

PROPOSITION 3.29. *Let X be a self-distributive BE-algebra and let d be a f -derivation of X . Then $F_a = \{x \in X \mid a \leq d(x)\}$ is a normal filter of X .*

Proof. Clearly, $a \leq d(1) = 1$ for any $a \in X$, and so $1 \in F_a$. Let $x \in X$ and $y \in F_a$. Then we have $a * d(y) = 1$, and so from Theorem 3.10,

$$\begin{aligned} a * d(x * y) &= a * (f(x) * d(y)) = (a * f(x)) * (a * d(y)) \\ &= (a * f(x)) * 1 \\ &= 1, \end{aligned}$$

which implies $x * y \in F_a$. Hence F_a is a normal filter of X . \square

DEFINITION 3.30. Let f be a map on X . A filter F of a BE-algebra X is said to be f -filter if $f(F) \subseteq F$.

DEFINITION 3.31. Let d be a self-map of a BE-algebra X . A f -filter F of X is said to be a d -invariant if $d(F) \subseteq F$.

PROPOSITION 3.32. *Let X be a self-distributive BE-algebra X . If d is a f -derivation of X , then every f -filter F is d -invariant.*

Proof. Let F be a f -filter of X . Let $y \in d(F)$. Then $y = d(x)$ for some $x \in F$. It follows from Proposition 3.9(1) that $f(x) * y = f(x) * d(x) = 1 \in F$. Since F is a f -filter of X , we have $y \in F$. Thus $d(F) \subseteq F$. Hence F is d -invariant. \square

References

- [1] A. Firat, *On f -derivations of BCC-algebras*, Ars Combinatoria, **XCVIIA** (2010), 377-382.
- [2] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313-320.
- [3] Q. P. Hu and X. Li, *On proper BCH-algebras*, Math Japonicae **30** (1985), 659-661.
- [4] K. Iseki and S. Tanaka, *An introduction to theory of BCK-algebras*, Math. Japonicae **23** (1978), 1-20.
- [5] K. Iseki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125-130.
- [6] K. H. Kim and S. M. Lee, *On derivations of BE-algebras*, Honam Mathematical Journal **36** (2014), no. 1, 167-178.
- [7] S. S. Ahn and K. S. So, *On ideals and uppers in BE-algebras*, Sci. Math. Japo. Online e-2008, 351-357.

*

Department of Mathematics
Korea National University of Transportation
Chungju 380-702, Republic of Korea
E-mail: ghkim@ut.ac.kr

**

Department of Mathematics,
Yazd University
Yazd, Iran
E-mail: davvaz@yazd.ac.ir